DYNAMICAL DIFFERENTIAL EQUATIONS COMPATIBLE WITH RATIONAL QKZ EQUATIONS

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March 2004

ABSTRACT. For the Lie algebra $\mathfrak{gl}_N$ we introduce a system of differential operators called the dynamical operators. We prove that the dynamical differential operators commute with the $\mathfrak{gl}_N$ rational quantized Knizhnik-Zamolodchikov difference operators. We describe the transformations of the dynamical operators under the natural action of the $\mathfrak{gl}_N$ Weyl group.

1. Introduction

The rational quantized Knizhnik-Zamolodchikov difference equations associated with the Lie algebra $\mathfrak{gl}_N$ is a system of difference equations of the form

$$U(z_1,\dots,z_i+p,\dots,z_n;\lambda_1,\dots,\lambda_N) = \\ = K_i(z_1,\dots,z_n;\lambda_1,\dots,\lambda_N)\,U(z_1,\dots,z_n;\lambda_1,\dots,\lambda_N)\,, \tag{1.1}$$

$i=1,\dots,n$. Here p is the step of the difference equations, $U(z;\lambda)$ is a function with values in the tensor product $V_1\otimes\ldots\otimes V_n$ of n highest weight $\mathfrak{gl}_N$-modules, $K_i(z;\lambda)$ is a suitable linear operator on the tensor product. The qKZ difference equations have found many applications; for example, see [EFK], [JM], [V2]. In [TV5] we suggested a system of differential equations of the form

$$\Bigl(p\,\lambda_a\frac{\partial}{\partial\lambda_a} + L_a(z_1,\dots,z_n;\lambda_1,\dots,\lambda_N)\Bigr)\,U(z_1,\dots,z_n;\lambda_1,\dots,\lambda_N) = 0\,, \tag{1.2}$$

$a=1,\dots,N$. Here $L_a(z;\lambda)$ is a suitable linear operator on $V_1\otimes\ldots\otimes V_n$. We called

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this system the dynamical differential equations. In this paper we prove that the qKZ difference equations and dynamical differential equations are compatible, and describe the transformation properties of the dynamical equations under the natural action of the $\mathfrak{gl}_N$ Weyl group.

The phenomenon of existence of dynamical equations compatible with KZ equations was discovered in [FMTV]. There are many versions of KZ equations: rational and trigonometric, differential and difference. For each version of the KZ equations there exists a complementary system of dynamical equations which is compatible with the KZ equations. In [FMTV] the rational differential KZ equations were considered and the compatible dynamical differential equations were introduced. In [TV4] the trigonometric differential KZ equations were considered and the compatible dynamical difference equations were introduced. In [EV] the trigonometric difference KZ equations were considered and the compatible dynamical difference equations were introduced.

All versions of KZ equations have hypergeometric solutions, see [SV], [V1], [TV1], [TV2], [TV3], [FV], [FTV]. The general conjecture is that the hypergeometric solutions also satisfy the corresponding dynamical equations. For rational KZ differential equations that was proved in [FMTV], for trigonometric KZ differential equations that was proved in [MV]. We plan to prove that the hypergeometric solutions of the rational qKZ difference equations (1.1) also satisfy the dynamical differential equations (1.2) in our next paper.

The fact that the hypergeometric solutions of the KZ equations satisfy also the additional dynamical equations is useful for aplications. For example, the dynamical equations, in principal, allow us to recover the hypergeometric solution from the asymptotics of the solution as λ tends to a special value. In [TV6] we used the dynamical equations in that way to find a formula for Selberg type integrals associated with $\mathfrak{sl}_3$.

There is also another phenomenon: the KZ and dynamical equations correspond to each other under the $(\mathfrak{gl}_N, \mathfrak{gl}_n)$ duality. This phenomenon was discovered in [T] and [TV5]. It turns out that under the $(\mathfrak{gl}_N, \mathfrak{gl}_n)$ duality the KZ and dynamical equations associated with $\mathfrak{gl}_N$ become respectively the dynamical and KZ equations associated with $\mathfrak{gl}_n$. That kind of the duality and the (partially conjectural) fact that hypergeometric solutions satisfy both the KZ and dynamical equations, in principal, allows us to identify hypergeometric solutions of the $\mathfrak{gl}_N$ and $\mathfrak{gl}_n$ KZ equations. We used that idea in [TV7] and [TV8] to prove certain nontrivial identities between hypergeometric integrals of different dimensions.

2. qKZ and dynamical operators

2.1. **The Yangian $Y(\mathfrak{gl}_N)$ and the rational R-matrix.** Let $e_{a,b}$, $a,b = 1,\dots N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$, $[e_{a,b}\,,\,e_{c,d}] = \delta_{bc}\,e_{a,d} - \delta_{ad}\,e_{c,b}$.

The Yangian $Y(\mathfrak{gl}_N)$ is the unital associative algebra with generators $T^{(s)}_{a,b}$ where $a,b = 1,\dots,N$ and $s = 1,2,\dots$. Organize them into generating series

$$T_{a,b}(u) \,=\, \delta_{a,b} + \sum_{s=1}^{\infty} T^{(s)}_{a,b} u^{-s}\,.$$

The defining relations in $Y(\mathfrak{gl}_N)$ have the form

$$\bigl[T_{a,b}(u)\,,T_{c,d}(v)\bigr] \,=\, \frac{T_{c,b}(v)\,T_{a,d}(u) - T_{c,b}(u)\,T_{a,d}(v)}{u-v}\,. \tag{2.1}$$

The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra with coproduct $\Delta : Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$,

$$\Delta : T_{a,b}(u) \mapsto \sum_{c=1}^{N} T_{c,b}(u) \otimes T_{a,c}(u) .$$

There is a one-parametric family of automorphism $\rho_x : Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N)$,

$$\rho_x : T_{a,b}(u) \mapsto T_{a,b}(u-x) .$$

The Yangian $Y(\mathfrak{gl}_N)$ contains the universal enveloping algebra $U(\mathfrak{gl}_N)$ as a Hopf subalgebra. The embedding is defined by $e_{a,b} \mapsto T^{(1)}_{b,a}$ for all $a,b = 1,\dots,N$. We identify $U(\mathfrak{gl}_N)$ with its image in $Y(\mathfrak{gl}_N)$ under this embedding.

There is an evaluation homomorphism $\epsilon : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$,

$$\epsilon : T_{a,b}(u) \mapsto \delta_{a,b} + e_{b,a} u^{-1}. \tag{2.2}$$

Both the automorphism ρ_x and the homomorphism ϵ restricted to the subalgebra $U(\mathfrak{gl}_N)$ are the identity maps.

For a $\mathfrak{gl}_N$-module V denote by $V(x)$ the $Y(\mathfrak{gl}_N)$-module induced from V by the homomorphism $\epsilon \circ \rho_x$. The module $V(x)$ is called an evaluation module.

Let V_1, V_2 be Verma modules over $\mathfrak{gl}_N$ with highest weight vectors v_1, v_2, respectively. For generic complex numbers x, y the Yangian modules $V_1(x) \otimes V_2(y)$ and $V_2(y) \otimes V_1(x)$ are known to be isomorphic. An isomorphism of the modules sends $\mathbb{C}(v_1 \otimes v_2)$ to $\mathbb{C}(v_2 \otimes v_1)$. We fix an isomorphism requiring that the vector $v_1 \otimes v_2$ is mapped to $v_2 \otimes v_1$. The isomorphism has the form

$$P R_{V_1,V_2}(x-y) : V_1(x) \otimes V_2(y) \to V_2(y) \otimes V_1(x)$$

where $P : V_1 \otimes V_2 \to V_2 \otimes V_1$ is the permutation of factors, and $R_{V_1,V_2}(x)$ takes values in $\mathrm{End}(V_1 \otimes V_2)$. The operator $R_{V_1,V_2}(x)$ respects the weight decomposition of the $\mathfrak{gl}_N$ module $V_1 \otimes V_2$ and its restriction to any weight subspace is a rational function of x. The operator $R_{V_1,V_2}(x)$ is called the rational R-matrix for the tensor product $V_1 \otimes V_2$.

The definition of $R_{V_1,V_2}(x)$ as a normalized intertwiner of the tensor product of evaluation modules is equivalent to the following relations:

$$R_{V_1,V_2}(x)\, v_1 \otimes v_2 = v_1 \otimes v_2 , \tag{2.3}$$

$$\bigl[R_{V_1,V_2}(x) , e_{a,b} \otimes \mathrm{id} + \mathrm{id} \otimes e_{a,b} \bigr] = 0 ,$$

$$R_{V_1,V_2}(x) \bigl(x\, \mathrm{id} \otimes e_{a,b} - \sum_{c=1}^{N} e_{a,c} \otimes e_{c,b} \bigr) = \bigl(x\, \mathrm{id} \otimes e_{a,b} - \sum_{c=1}^{N} e_{c,b} \otimes e_{a,c} \bigr) R_{V_1,V_2}(x) ,$$

for any $a, b = 1, , \dots, N$.

Let V_1, V_2, V_3 be Verma modules over $\mathfrak{gl}_N$. The corresponding R-matrices satisfy the Yang-Baxter equation:

$$R^{(1,2)}_{V_1,V_2}(x-y)\, R^{(1,3)}_{V_1,V_3}(x)\, R^{(2,3)}_{V_2,V_3}(y) = R^{(2,3)}_{V_2,V_3}(y)\, R^{(1,3)}_{V_1,V_3}(x)\, R^{(1,2)}_{V_1,V_2}(x-y) .$$

The formulated facts on the Yangian are well known; for example, see [MNO].

2.2. **The qKZ and dynamical operators associated with $\mathfrak{gl}_N$.** Let $V_1,\dots,V_n$ be Verma modules over $\mathfrak{gl}_N$. Let $R_{V_i,V_j}(x)$ be the corresponding rational R-matrices. Let $p,\ \lambda_1,\dots,\lambda_N$ be nonzero complex numbers. Denote by T_u the difference operator acting on a function $f(u)$ by the formula

$$(T_u f)(u)\,=\,f(u+p)\,.$$

Define the operators $K_1,\dots,K_n$ acting on $V_1\otimes\ldots\otimes V_n$:

$$K_m(z;\lambda)\,=\,\bigl(R^{(1,m)}_{V_1,V_m}(z_1-z_m-p)\ldots R^{(m-1,m)}_{V_{m-1},V_m}(z_{m-1}-z_m-p)\bigr)^{-1}\times$$
$$\times\prod_{a=1}^{N}\lambda_a^{e^{(m)}_{a,a}}\,R^{(m,n)}_{V_m,V_n}(z_m-z_n)\ldots R^{(m,m+1)}_{V_m,V_{m+1}}(z_m-z_{m+1})\,.$$

Introduce the difference operators $Z_1,\dots,Z_n$, $\quad Z_i=\bigl(K_i(z;\lambda)\bigr)^{-1}T_{z_i}$, called the qKZ operators. They act on $V_1\otimes\ldots\otimes V_n$-valued functions of $z_1,\dots,z_n,\ \lambda_1,\dots,\lambda_N$.

Theorem 1 ([FR]). *The qKZ operators $Z_1,\dots,Z_n$ pairwise commute. In other words,*

$$K_l(z_1,\dots,z_m+p,\dots,z_n;\lambda)\,K_m(z_1,\dots,z_n;\lambda)\,=$$
$$=\,K_m(z_1,\dots,z_l+p,\dots,z_n;\lambda)\,K_l(z_1,\dots,z_n;\lambda)$$

for all $l,m=1,\dots,n$.

For $a,b=1,\dots,N$, the element $e_{a,b}$ acts on $V_1\otimes\ldots\otimes V_n$ as $\sum_{i=1}^{n}e^{(i)}_{a,b}$. Define the operators $L_1,\dots,L_N$ acting on $V_1\otimes\ldots\otimes V_n$:

$$L_a(z;\lambda)\,=\,\frac{(e_{a,a})^2}{2}\,-\,\sum_{i=1}^{n}z_i\,e^{(i)}_{a,a}\,-\,\sum_{b=1}^{N}\sum_{1\leqslant i<j\leqslant n}e^{(i)}_{a,b}\,e^{(j)}_{b,a}\,-\,\sum_{\substack{b=1\\ b\neq a}}^{N}\frac{\lambda_b}{\lambda_a-\lambda_b}\,(e_{a,b}\,e_{b,a}-e_{a,a})\,.$$

Introduce the differential operators $D_1,\dots,D_N$,

$$D_a\,=\,p\,\lambda_a\frac{\partial}{\partial\lambda_a}\,+\,L_a(z;\lambda)\,,$$

called the dynamical operators. They act on $V_1\otimes\ldots\otimes V_n$-valued functions of $z_1,\dots,$ $z_n,\ \lambda_1,\dots,\lambda_N$.

Theorem 2 ([TV5]). *The dynamical operators $D_1,\dots,D_N$ pairwise commute.*

The theorem is proved by direct verification.

2.3. **Compatibility of qKZ and dynamical operators.**

Theorem 3. *The qKZ operators $Z_1,\dots,Z_n$ commute with the dynamical operators $D_1,$ $\dots,D_N$.*

Proof. Since the operators $Z_1,\dots,Z_n$ are invertible and pairwise commute, the claim of the theorem is equivalent to the following identities:

$$[Z_1\ldots Z_i\,,D_a]\,=0$$

for all $a=1,\dots,N$ and $i=1,\dots,n$. To simplify notations we consider the case $i=1$. The general case is similar.

A simple transformation converts identity $[Z_1, D_a] = 0$ into identity

$$\bigl[K_1(z;\lambda), L_a(z;\lambda)\bigr] = 0\,.$$

Here $K_1(z;\lambda) = \prod_{a=1}^{N} \lambda_a^{e_{a,a}^{(1)}} R^{(1,n)}_{V_1,V_n}(z_1-z_n)\dots R^{(1,2)}_{V_1,V_2}(z_1-z_2)$. We have

$$L_a(z;\lambda) = \frac{(e_{a,a})^2}{2} - \sum_{\substack{b=1\\ b\neq a}}^{N} \frac{\lambda_b}{\lambda_a-\lambda_b}\,(e_{a,b}\,e_{b,a} - e_{a,a}) - z_1 e_{a,a} + {} \\ + \sum_{i=2}^{n}(z_1-z_i)\,e^{(i)}_{a,a} - \sum_{b=1}^{N}\sum_{1\leqslant i<j\leqslant n} e^{(i)}_{a,b}\,e^{(j)}_{b,a}\,. \tag{2.4}$$

The term $z_1 e_{a,a}$ commutes with $K_1(z;\lambda)$. The first two terms in the right hand side of (2.4) commute with the product $R^{(1,n)}_{V_1,V_n}(z_1-z_n)\dots R^{(1,2)}_{V_1,V_2}(z_1-z_2)$ since the R-matrices commute with coproducts. To proceed with the last two terms we are using the second and third relations in (2.3), and commutativity of $R^{(1,i)}_{V_1,V_i}$ and $e^{(j)}_{c,d}$ for distinct i,j not equal to 1. For example,

$$R^{(1,2)}_{V_1,V_2}(z_1-z_2)\Bigl((z_1-z_2)\,e^{(2)}_{a,a} - \sum_{b=1}^{N} e^{(1)}_{a,b}\,e^{(2)}_{b,a} + \sum_{i=3}^{n}(z_1-z_i)\,e^{(i)}_{a,a} - {} \\ - \sum_{b=1}^{N}\sum_{j=3}^{n}(e^{(1)}_{a,b}+e^{(2)}_{a,b})\,e^{(j)}_{b,a} - \sum_{b=1}^{N}\sum_{3\leqslant i<j\leqslant n} e^{(i)}_{a,b}\,e^{(j)}_{b,a}\Bigr) = {} \\ = \Bigl((z_1-z_2)\,e^{(2)}_{a,a} - \sum_{b=1}^{N} e^{(1)}_{b,a}\,e^{(2)}_{a,b} + \sum_{i=3}^{n}(z_1-z_i)\,e^{(i)}_{a,a} - {} \\ - \sum_{b=1}^{N}\sum_{j=3}^{n}(e^{(1)}_{a,b}+e^{(2)}_{a,b})\,e^{(j)}_{b,a} - \sum_{b=1}^{N}\sum_{3\leqslant i<j\leqslant n} e^{(i)}_{a,b}\,e^{(j)}_{b,a}\Bigr)\,R^{(1,2)}_{V_1,V_2}(z_1-z_2)\,,$$

and consecutive calculations with other R-matrices $R^{(1,i)}_{V_1,V_i}$ are similar. Finally, we obtain

$$K_1(z;\lambda)\,L_a(z;\lambda) = -z_1 e_{a,a}\,K_1(z;\lambda) + {} \\ + \prod_{c=1}^{N}\lambda_c^{e^{(1)}_{c,c}}\Bigl(\frac{(e_{a,a})^2}{2} - \sum_{\substack{b=1\\ b\neq a}}^{N}\frac{\lambda_b}{\lambda_a-\lambda_b}\,(e_{a,b}\,e_{b,a}-e_{a,a}) + {} \\ + \sum_{i=2}^{n}(z_1-z_i)\,e^{(i)}_{a,a} - \sum_{b=1}^{N}\sum_{j=2}^{n} e^{(1)}_{b,a}\,e^{(j)}_{a,b} - \sum_{b=1}^{N}\sum_{2\leqslant i<j\leqslant n} e^{(i)}_{a,b}\,e^{(j)}_{b,a}\Bigr)\times \\ \times\, R^{(1,n)}_{V_1,V_n}(z_1-z_n)\dots R^{(1,2)}_{V_1,V_2}(z_1-z_2)\,. \tag{2.5}$$

The right hand side of (2.5) can be transformed into $L_a(z;\lambda)\,K_1(z;\lambda)$ in a straightforward way. Theorem 3 is proved. □

2.4. **Dynamical operators and the Weyl group.** The symmetric group S_N is the Weyl group for the Lie algebra $\mathfrak{gl}_N$. There is a natural action, denoted by π, of S_N on $U(\mathfrak{gl}_N)$: $\pi(w)(e_{a,b}) = e_{w(a),w(b)}$ for $w \in S_N$ and $a, b = 1, \ldots, N$.

Consider the space of linear differential operators of the form

$$D \,=\, p\lambda_a \frac{\partial}{\partial\lambda_a} \,+\, M(z;\lambda)\,,$$

where $a \in \{1, \ldots, N\}$ and M is a $U(\mathfrak{gl}_N)$-valued function of $z_1, \ldots, z_n$, $\lambda_1, \ldots, \lambda_N$. The group S_N acts on this space by the formula

$$wD \,=\, p\lambda_{w(a)} \frac{\partial}{\partial\lambda_{w(a)}} \,+\, \pi(w)M(z;\lambda_{w(1)}, \ldots, \lambda_{w(N)})\,.$$

Theorem 4. *We have* $wD_a = D_{w(a)}$ *for all* $w \in S_N$ *and* $a = 1, \ldots, N$.

The proof is straightfirward.

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